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A CHARACTERIZATION OF CARDINALS κ SUCH THAT $2^\lambda = 2^\kappa$ WHENEVER $\kappa \leqq \lambda < 2^\kappa$

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ABSTRACT

We characterize cardinals κ such that $2^\lambda = 2^\kappa$ whenever $\kappa \leqq \lambda < 2^\kappa$ using ideals in small algebras of sets satisfying certain completeness and saturation conditions.

0. Introduction

If 2^ω is real-valued measurable or, more generally, carries a 2^ω-complete σ-saturated nontrivial ideal, then $2^\omega = 2^\lambda$ for all infinite cardinals $\lambda < 2^\omega$. A detailed proof of this result of [P] can be found in [J, Proof of Theorem 83, p. 426]. We observe that under the hypothesis of the above stated theorem, 2^ω is a very large cardinal (see [U] and [S_1]) and thus the equation $2^\omega = 2^\lambda$ is satisfied by many cardinals λ. In this paper we generalize the above mentioned theorem. In Theorem 1, we give some very weak conditions on a cardinal λ entailing the equation $2^\lambda = 2^{\underset{\smile}{\lambda}}$ (where $2^{\underset{\smile}{\lambda}} = \sup\{2^\nu : \nu < \lambda\}$). This theorem is then used to obtain, in Theorem 2, several characterizations of cardinals κ satisfying the condition in the title. The result of [P] mentioned above is obtained in Corollaries 1(i) and 2(i).

The implications for cardinal exponentiation of the existence of ideals, with strong completeness and saturation properties, have also been investigated in [KT], [S_2] and [JP], among others. While these earlier works deal with ideals in the entire power set, in this paper we consider ideals in small subalgebras of the power set algebra. This latter approach has also been adopted in [Ca] and [PP].

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It leads to demonstrably weaker conditions — ones which do not entail the existence of large cardinal numbers. In fact, our conditions, following as they do from the condition in the title, are (nontrivially) satisfied in Cohen's models [Co] of the negation of the continuum hypothesis and, in an even more general form, in Easton-type models [E].

Finally, we point out that the method of [P] uses a result from partition calculus, whereas the present approach is motivated by measure-theoretic considerations.

1. Notation, definitions and statement of the results

Our set theoretic notation is standard. The power set and the cardinality of a set A are denoted by $\mathscr{P}(A)$ and $|A|$, respectively. If κ is a cardinal then κ^+ and $\mathrm{cf}(\kappa)$ denote the successor of κ, and the cofinality of κ, respectively. Furthermore $2^{\underset{\smile}{\kappa}}$ is the supremum of 2^λ, $\lambda < \kappa$. Each ordinal is identified with the set of smaller ordinals and cardinals are initial ordinals. If f is a function and E is contained in the domain of f, then $f \restriction E$ is the restriction of f to E.

An algebra on X is a subalgebra of $\mathscr{P}(X)$. Let $\mathscr{A}$ be an algebra on X. $\mathscr{A}$ is generated by $\mathscr{F}$ if $\mathscr{A}$ is the least algebra on X such that $\mathscr{F} \subseteq \mathscr{A}$. Let $\mathscr{I}$ be an ideal in $\mathscr{A}$. $\mathscr{I}$ is *nontrivial* if $\mathscr{I}$ is proper and contains all singletons of $\mathscr{A}$. $\mathscr{I}$ is *κ-complete* if for every $\mathscr{I}^* \subseteq \mathscr{I}$ with $|\mathscr{I}^*| < \kappa$ there is some $A \in \mathscr{I}$ such that $\bigcup \mathscr{I}^* \subseteq A$. $\mathscr{I}$ is *weakly κ-complete* if for every $\mathscr{I}^* \subseteq \mathscr{I}$ with $|\mathscr{I}^*| < \kappa$, $\bigcup \mathscr{I}^* \neq X$. A family $\mathscr{A}^* \subseteq \mathscr{A}$ is *$\mathscr{I}$-disjoint* if for all $A_1, A_2 \in \mathscr{A}^*$ such that $A_1 \neq A_2$ we have $A_1 \cap A_2 \in \mathscr{I}$. $\mathscr{I}$ is *κ-saturated* (resp. *weakly κ-saturated*) if $|\mathscr{A}^*| < \kappa$ for every $\mathscr{I}$-disjoint (resp. disjoint) family $\mathscr{A}^* \subseteq \mathscr{A} \setminus \mathscr{I}$. $\mathscr{A}$ is called a *κ-algebra* if $\mathscr{A}$ is closed under unions of subfamilies of cardinality less than κ. $\mathscr{A}$ is *κ-generated* by $\mathscr{F}$ if $\mathscr{A}$ is the least κ-algebra on X such that $\mathscr{F} \subseteq \mathscr{A}$.

Let $\mathscr{A}$ be any family of sets, S a set and κ a cardinal number. $\mathscr{A}$ is said to be *κ-saturated with respect to S* if there is no $\mathscr{A}^* \subseteq \mathscr{A}$, consisting of pairwise disjoint sets all of which have nonempty intersection with S, and $|\mathscr{A}^*| = \kappa$.

We say that (i) ρ has the *weak (τ, λ)-extension property* if every algebra on ρ generated by the singletons and at most τ additional sets has a weakly τ^+-complete weakly λ-saturated nontrivial ideal; (ii) ρ has the *(τ, λ)-coherence property* if for every family $\mathscr{A} \subseteq \mathscr{P}(\rho)$ such that $|\mathscr{A}| \leqq \tau$ there is an $S \subseteq \rho$ and $x_0 \in S$ such that $\mathscr{A}$ is λ-saturated with respect to S and for all $A \in \mathscr{A}$, if $x_0 \in A$, then $|A \cap S| \geqq 2$. We shall see in Section 2, Lemma 1 that if ρ has the weak (τ, λ)-extension property then ρ has the (τ, λ)-coherence property.

A cardinal ρ is *real-valued measurable* if there is a ρ-additive probability measure on $\mathscr{P}(\rho)$ vanishing on all singletons.

We shall now state our results and derive Corollaries 1 and 2 from Theorem 2. Proofs of Theorems 1 and 2 are given in the following section.

THEOREM 1. *Let λ be an infinite regular cardinal and* $\mathrm{cf}(\nu) > \lambda$ *where* $\nu = 2^{\underset{\sim}{\lambda}}$. *Also suppose that ν^+ has the (τ, λ)-coherence property (or the weak (τ, λ)-extension property) for every cardinal $\tau < \nu$ such that* $\mathrm{cf}(\tau) = \lambda$. *Then $2^\lambda = \nu$.*

THEOREM 2. *Let κ be an infinite cardinal and set $\nu = 2^\kappa$. Then the following are equivalent*:

(a) *$2^\lambda = \nu$ for all λ such that $\kappa < \lambda < \nu$;*

(b) *ν is regular and ν^+ has the (τ, κ^+)-coherence property for all $\tau < \nu$;*

(c) *ν is regular and ν^+ has the $(\tau,$ $\mathrm{cf}(\tau))$-coherence property for all $\tau < \nu$ such that* $\mathrm{cf}(\tau) > \kappa$;

(d) *ν is regular and ν^+ has the weak (τ, κ^+)-extension property for every cardinal $\tau < \nu$;*

(e) *ν is regular and ν^+ has the weak $(\tau,$ $\mathrm{cf}(\tau))$-extension property for every cardinal $\tau < \nu$ such that* $\mathrm{cf}(\tau) > \kappa$;

(f) *every ν^+-algebra on ν^+, ν^+-generated by the singletons and less than ν additional sets has a ν^+-complete nontrivial prime ideal.*

REMARK. We don't know if the regularity of ν in conditions (b)–(e) above follows from the other half of each of these conditions.

COROLLARY 1. *Let κ be an infinite cardinal and set $\nu = 2^\kappa$. Suppose that one of the following conditions* (i) *or, more generally,* (ii) *holds*:

(i) *There is a ν-complete κ^+-saturated nontrivial ideal in $\mathscr{P}(\nu)$.*

(ii) *ν is regular and for every cardinal $\tau < \nu$ there is some ρ such that $\tau < \rho \leqq \nu$ and there is a τ^+-complete κ^+-saturated nontrivial ideal $\mathscr{I}$ in $\mathscr{P}(\rho)$.*

Then $2^\lambda = \nu$ for all λ such that $\kappa \leqq \lambda < \nu$.

COROLLARY 2. *Set $\nu = 2^\omega$ and suppose that one of the following conditions* (i) *or, more generally,* (ii) *holds*:

(i) *ν is real-valued measurable.*

(ii) *ν is regular and for every $\tau < \nu$ there is a real-valued measurable cardinal ρ such that $\tau < \rho \leqq \nu$.*

Then $2^\lambda = \nu$ for all infinite $\lambda < \nu$.

PROOF OF COROLLARY 1. We shall show that both (i) and (ii) imply (d) of

Theorem 2. If (i) holds, ν is regular by [S_1, Section 2, Lemma 2]. Now let $\mathscr{A} \subseteq \mathscr{P}(\nu^+)$, where $|\mathscr{A}| \leqq \tau < \nu$. Using either (i) or (ii) we can pick some ρ such that $\tau < \rho \leqq \nu$ and $\mathscr{P}(\rho)$ has a τ^+-complete κ^+-saturated nontrivial ideal $\mathscr{I}$. Let $\tilde{\mathscr{A}}$ be the algebra on ν^+ generated by $\mathscr{A}$ and the singletons. We now define an ideal $\mathscr{J}$ in $\tilde{\mathscr{A}}$ by $\mathscr{A} \in \mathscr{J}$ iff $\mathscr{A} \cap \rho \in \mathscr{I}$. It is easy to verify that $\mathscr{J}$ satisfies the conditions of (d). Thus, since (d) implies (a), the conclusion of Corollary 1 follows.

PROOF OF COROLLARY 2. Corollary 2 follows immediately from Corollary 1 since if ρ is a real-valued measurable cardinal then the null ideal of a probability measure on $\mathscr{P}(\rho)$, witnessing to the real-valued measurability of ρ, is a ρ-complete ω_1-saturated nontrivial ideal.

We shall conclude this section with a remark for the expert reader outlining our key argument. In particular, we sketch a proof that if $2^\kappa = \nu$ and there is a ν-complete κ^+-saturated nontrivial ideal in $\mathscr{P}(\nu)$, then $2^\lambda = \nu$ for all λ such that $\kappa < \lambda < \nu$. Our terminology is the same as in [B] or [J].

Suppose $\lambda < \nu$ is minimal with $2^\lambda > \nu$. Then λ is regular and $\lambda > \kappa$. Find a tree T of height λ so that $|T| < \nu$ and T has at least ν branches of length λ (so far this is like the argument in [J, Proof of Theorem 83, p. 426]). For $t \in T$, let S_t be the set of all branches of T containing t. Put the ideal given by the hypothesis on the set of all branches of T, and let N be the union of all sets S_t which belong to the ideal. Then N belongs to the ideal. If b is a branch not in N then note that $\{S_t : t \in b\}$ contains a sequence of λ properly decreasing sets not in the ideal, contradicting κ^+-saturation.

The crucial observation is that there is no need whatever for the ideal to be defined on the full power set of ν; it is only necessary that it be defined on a collection (not even necessarily an algebra) containing the sets S_t. Moreover, since T could be chosen with ν^+ branches we could assume the S_t are subsets of ν^+, not ν. This now points the way to working with ideals in small algebras of sets as well as to some more purely combinatorial results.

2. Proofs of Theorems 1 and 2

LEMMA 1. *If ρ has the weak (τ,λ)-extension property then ρ has the (τ,λ)-coherence property.*

PROOF. Let $\mathscr{A} \subseteq \mathscr{P}(\rho)$ where $|\mathscr{A}| \leqq \tau$. Let $\mathscr{A}$ be the algebra on ρ generated by $\mathscr{A}$ and the singletons. Let $\tilde{\mathscr{I}}$ be a weakly τ^+-complete weakly λ-saturated nontrivial ideal in $\mathscr{A}$, as guaranteed by the hypothesis. Let $\mathscr{I}$ be the ideal in $\mathscr{P}(\rho)$ consisting of those sets which can be covered by $\leqq \tau$ sets in $\tilde{\mathscr{I}}$. Obviously, $\mathscr{I}$ is τ^+-complete and nontrivial (ρ cannot belong to $\mathscr{I}$ by the weak τ^+-completeness of $\tilde{\mathscr{I}}$).

Now set $N = \bigcup(\mathscr{I} \cap \mathscr{A})$, $S = \rho \setminus N$. Since $|\mathscr{A}| \leqq \tau$, we have $N \in \mathscr{I}$. Hence $S \neq \emptyset$.

CLAIM. If $A \in \mathscr{A}$ and $A \cap S \neq \emptyset$ then $A \cap S \notin \mathscr{I}$ and thus $|A \cap S| \geqq 2$.

Indeed, if $A \cap S \in \mathscr{I}$ then $N \cup (A \cap S) \in \mathscr{I}$ and thus $A \in \mathscr{I}$; in particular, $A \subseteq N$, contradicting $A \cap S \neq \emptyset$.

We can now show that $\mathscr{A}$ is λ-saturated with respect to S. To this end, let $\mathscr{A}^* \subseteq \mathscr{A}$ consist of pairwise disjoint sets all of which meet S, and let $|\mathscr{A}^*| \geqq \lambda$.

Choose an $A \in \mathscr{A}^*$. Then $A \in \mathscr{A}$ and $A \cap S \neq \emptyset$, so by the Claim, $A \cap S \notin \mathscr{I}$ and thus $A \notin \mathscr{I}$. Hence $A \notin \tilde{\mathscr{I}}$ since $\tilde{\mathscr{I}} \subseteq \mathscr{I}$. Consequently $\mathscr{A}^*$ witnesses to $\tilde{\mathscr{I}}$ not being λ-saturated and we have reached a contradiction.

It is also clear from the Claim that for x_0 required in the (τ, λ)-coherence property, we can pick any element of S. This completes the proof of Lemma 1.

In the following Lemma 2, we investigate the least possible number of initial segments of members of a large subset of $\{0, 1\}^\lambda$. This lemma is a corollary of [B, Corollary 3.2, p. 412]. However, in order to keep the paper self-contained, we wish to include a proof of Lemma 2 which does not employ the somewhat specialized set-theoretic terminology of [B]; readers with a background in measure theory may appreciate this. For these readers, we will also indicate, in the concluding remark, a shortcut to the proofs of Corollaries 1 and 2 using only the easier part — $\mathrm{cf}(\rho) \leqq \lambda$ — of Lemma 2. The original applications of such results in [B] are to obtain large families of almost disjoint sets. Here the full strength of Lemma 2 is used to obtain the equivalences (c) and (e) in Theorem 2. Only the easy part is needed for the proof that (a), (b), (d) and (f) are equivalent.

LEMMA 2. *Let λ be an infinite regular cardinal and set $2^{\underset{\smile}{\lambda}} = \nu$. Suppose that $\nu < 2^\lambda$. Let ρ be minimal such that there is an $X \subseteq \{0, 1\}^\lambda$ with $|X| = \nu^+$ and $|Y| = \rho$, where*

$$Y = \{x \upharpoonright \alpha : x \in X, \alpha < \lambda\}.$$

Then $\rho \leqq \nu$ and $\mathrm{cf}(\rho) = \lambda$.

PROOF. Let X, Y and ρ be as above. Firstly $\rho \leqq \nu$ since

$$Y \subseteq \bigcup\{\{0,1\}^\alpha : \alpha < \lambda\},$$

and the latter set has cardinality ν. Likewise $\lambda \leqq \rho$ since any one function $f \in X$ has exactly λ proper initial segments.

Now fix a bijection $g : Y \to \rho$.

We show first that $\mathrm{cf}(\rho) \leqq \lambda$. Suppose $\mathrm{cf}(\rho) > \lambda$. For each $x \in X$ set

$$\eta_x = \sup\{g(x \restriction \alpha) : \alpha < \lambda\}.$$

Then $\eta_x < \rho$ and thus, as $\rho \leqq \nu$, there is $X^* \subseteq X$ and $\eta < \rho$ such that $|X^*| = \nu^+$ and $\eta_x < \eta$ for all $x \in X^*$. Hence since g is an injection and

$$g(\{x \restriction \alpha : x \in X^*, \alpha < \lambda\}) \subseteq \eta,$$

we have

$$|\{x \restriction \alpha : x \in X^*, \alpha < \lambda\}| \leqq |\eta| < \rho,$$

contradicting the choice of X.

It remains to show that $\mathrm{cf}(\rho) \geqq \lambda$. If $\rho = \lambda$, we are done, for λ is regular. Thus in view of $\lambda \leqq \rho$ it now suffices to reach a contradiction from the assumption that $\mathrm{cf}(\rho) < \lambda < \rho$. Let R be a cofinal subset of ρ with $|R| = \mathrm{cf}(\rho)$. Since λ is regular, for each $x \in X$ we can pick a $\gamma_x \in R$ such that

$$L_x = \{\alpha \in \lambda : g(x \restriction \alpha) \leqq \gamma_x\}$$

has cardinality λ. Similarly we can pick a $\gamma^* < \rho$ such that

$$X^* = \{x \in X : \gamma_x < \gamma^*\}$$

has cardinality ν^+. Let

$$Y^* = \{x \restriction \alpha : x \in X^*, \alpha < \lambda\}.$$

We reach a contradiction by showing that $|Y^*| < \rho$. To do this we define an injection $h : Y^* \to \lambda \times \gamma^*$ as follows. Choose a $t \in Y^*$ of length α. Let $\beta \geqq \alpha$ be minimal such that for some $x \in X^*$, $\beta \in L_x$ and $x \restriction \alpha = t$. Pick such an x and set $h(t) = (\alpha, g(x \restriction \beta))$. Since g is an injection and $(x \restriction \beta) \restriction \alpha = t$, it is clear that h is an injection. Thus $|Y^*| \leqq \lambda \cdot |\gamma^*| < \rho$, a contradiction.

PROOF OF THEOREM 1. By Lemma 1 it suffices to prove the theorem assuming that ν^+ has the (τ, λ)-coherence property for every cardinal $\tau < \nu$ such that $\mathrm{cf}(\tau) = \lambda$. Proceeding towards a contradiction, assume that $2^\lambda > \nu$.

Choose X, Y and ρ as in Lemma 2. Thus $|X| = \nu^+$, $|Y| = \rho \leqq \nu$ and $\mathrm{cf}(\rho) = \lambda$. It follows that $\rho < \nu$; otherwise $\rho = \nu$ and thus $\mathrm{cf}(\nu) = \mathrm{cf}(\rho) = \lambda$ contradicting the assumption that $\mathrm{cf}(\nu) > \lambda$. Thus ν^+ has the (ρ, λ)-coherence property.

Let $\mathscr{A}$ be the algebra on X generated by the sets

$$X(t) = \{x \in X : x \text{ extends } t\} \qquad (t \in Y).$$

Then $|\mathscr{A}| = |Y| = \rho$. Since ν^+ has the (ρ, λ)-coherence property we can choose an $S \subseteq X$ and $x_0 \in S$ so that $\mathscr{A}$ is λ-saturated with respect to S and for all $A \in \mathscr{A}$ if $x_0 \in A$, then $|A \cap S| \geqq 2$.

Writing briefly $X(\alpha)$ instead of $X(x_0 \restriction \alpha)$, we shall establish the following claim.

Claim. There is an $\alpha_0 \in \lambda$ such that for each α with $\alpha_0 < \alpha < \lambda$ we have

$$X(\alpha) \cap S = X(\alpha_0) \cap S.$$

Suppose the Claim is false. Then since λ is regular, there is a strictly increasing sequence α_ξ, $\xi < \lambda$, such that

$$X(\alpha_{\xi+1}) \cap S \neq X(\alpha_\xi) \cap S.$$

Since the sets $X(\alpha_\xi)$ are decreasing we see that the family

$$\{X(\alpha_\xi) \setminus X(\alpha_{\xi+1}) : \xi \in \lambda\}$$

consists of pairwise disjoint sets all of which have nonempty intersection with S. Hence $\mathscr{A}$ is not λ-saturated with respect to S. This contradition establishes the Claim.

We are ready to conclude the proof of the theorem. Let α_0 be as in the Claim. As $x_0 \in X(\alpha_0)$, by the choice of S and x_0 there is an $x \in X(\alpha_0) \cap S$ distinct from x_0. Let $\alpha > \alpha_0$, $\alpha \in \lambda$, be such that $x \restriction \alpha \neq x_0 \restriction \alpha$. Then $x \notin X(\alpha)$ and thus $X(\alpha) \cap S \neq X(\alpha_0) \cap S$, contradicting the choice of α_0.

Proof of Theorem 2. "(c) implies (a)" follows easily from Theorem 1. To see this suppose that $\kappa < \lambda < \nu$ and, proceeding inductively, that $2^\mu = \nu$ whenever $\kappa < \mu < \lambda$. If λ is singular, then $\lambda = \sup\{\lambda_\xi : \xi < \mathrm{cf}(\lambda)\}$ where $\lambda_\xi < \lambda$, $\mathrm{cf}(\lambda) < \lambda$, and $\nu = 2^{\lambda_\xi}$. Thus

$$2^\lambda = \prod_{\xi \in \mathrm{cf}(\lambda)} 2^{\lambda_\xi} = \nu^{\mathrm{cf}(\lambda)} = 2^{\kappa \cdot \mathrm{cf}(\lambda)} = \nu.$$

Now suppose that λ is regular. By (c), ν is also regular and hence, as $\lambda < \nu$, we have $\mathrm{cf}(\nu) > \lambda$. Since $2^\mu = \nu$ for $\kappa \leqq \mu < \lambda$, we see that $\nu = 2^{\check{\lambda}}$. Finally, if $\tau < \nu$ and $\mathrm{cf}(\tau) = \lambda$ we conclude from (c) that ν^+ has the (τ, λ)-coherence property (since $\lambda > \kappa$). Hence the assumptions of Theorem 1 are verified and $2^\lambda = \nu$ follows, proving (a).

Lemma 1 shows that (e) implies (c) and (d) implies (b). Moreover, (b) trivially implies (c), and (d) trivially implies (e).

In the next two steps we show that (a) and (f) are equivalent.

To show that (a) implies (f) let $\mathscr{A}$ be a ν^+-algebra on ν^+, ν^+-generated by the singletons and a family $\mathscr{F} \subseteq \mathscr{A}$ such that $|\mathscr{F}| = \tau < \nu$. Pick $\mathscr{F}_1 \subseteq \mathscr{F}$ so that $|S| = \nu^+$ where

$$S = (\bigcap \mathscr{F}_1) \cap \bigcap \{\nu^+ \setminus A : A \in \mathscr{F} \setminus \mathscr{F}_1\}.$$

This is possible since $2^\tau \leqq \nu$. Set

$$\tilde{\mathscr{A}} = \{A \subseteq \nu^+ : |A \cap S| \leqq \nu \text{ or } |(\nu^+ \setminus A) \cap S| \leqq \nu\},$$

$\tilde{\mathscr{I}} = \{A \in \tilde{\mathscr{A}} : |A \cap S| \leqq \nu\}$ and $\mathscr{I} = \tilde{\mathscr{I}} \cap \mathscr{A}$. Then $\tilde{\mathscr{A}}$ is a ν^+-algebra on ν^+ containing all the generators of $\mathscr{A}$ and thus $\mathscr{A}$ itself. $\mathscr{I}$ is the desired ideal in $\mathscr{A}$ since $\tilde{\mathscr{I}}$ is easily seen to be a nontrivial ν^+-complete prime ideal in $\tilde{\mathscr{A}}$.

The proof that (f) implies (a) is merely Tarski's proof that there is no λ^+-complete free ultrafilter in $\mathscr{P}(2^\lambda)$. Indeed, suppose that $2^\lambda > \nu$ where $\kappa < \lambda < \nu$ and let $T \subseteq \{0, 1\}^\lambda$, $|T| = \nu^+$. Then if $\mathscr{A}$ is any λ^+-algebra on T such that the relative subbasic subsets of T belong to $\mathscr{A}$ (and there are only λ such sets), it follows that every λ^+-complete ultrafilter in $\mathscr{A}$ is fixed. For it contains, for each $\alpha \in \lambda$, one of the sets $\{t \in T : t(\alpha) = i\}$, where $i = 0$ or 1, and thus by the λ^+-completeness it also contains $\{s\}$ for some $s \in T$. Consequently (f) fails.

The final step, showing that (f) implies (d), completes the proof of Theorem 2. Firstly since (f) implies (a), ν is regular by Konig's Theorem [J, Corollary 3, p. 46]. Now let $\mathscr{A}$ be an algebra on ν^+ generated by the singletons and some $\mathscr{F} \subseteq \mathscr{A}$ with $|\mathscr{F}| = \tau < \nu$. Let $\tilde{\mathscr{A}}$ be the ν^+-algebra on ν^+, ν^+-generated by $\mathscr{F}$ and the singletons. Let $\tilde{\mathscr{I}}$ be a ν^+-complete nontrivial prime ideal in $\tilde{\mathscr{A}}$ as guaranteed by (f). Then $\mathscr{I} = \tilde{\mathscr{I}} \cap \mathscr{A}$ is the desired ideal in $\mathscr{A}$.

REMARK. We shall now indicate a shortcut to the proofs of Corollaries 1 and 2 using only the easier direction — $\mathrm{cf}(\rho) \leqq \lambda$ — of Lemma 2. This direction suffices to establish Theorem 1* which is exactly like Theorem 1 except that the (τ, λ)-coherence property of ν^+ is now assumed for all $\tau < \nu$ — not just for those of cofinality λ. One can proceed exactly as in the proof of

Theorem 1 except that to conclude that ν^+ has the (ρ, λ)-coherence property, one now needs to know only that $\rho < \nu$. To establish that $\rho < \nu$ it clearly suffices to know only that $\mathrm{cf}(\rho) \leqq \lambda$.

Theorem 1* suffices to establish "(d) implies (a)" of Theorem 2 — the only part of Theorem 2 which is used in the proofs of Corollaries 1 and 2. Firstly "(d) implies (b)" follows from Lemma 1, as pointed out in the proof of Theorem 2. Finally, the proof, as given above, of "(c) implies (a)", which utilizes Theorem 1, is easily modified to give "(b) implies (a)" using Theorem 1* only.

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